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STRONGLY PERFECT INFINITE GRAPHS*

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ABSTRACT

An infinite graph G is called **strongly perfect** if each induced subgraph of G has a coloring (C_i: $i \in I$) and a clique meeting each color C_i. A conjecture of the first author and V. Korman is that a perfect graph with no infinite independent set is strongly perfect. We prove the conjecture for chordal graphs and for their complements.

1. Introduction

A (possibly infinite) graph G is called **perfect** if $\chi(H) = \omega(H)$ for each induced subgraph H of G, where $\chi(H)$ is the chromatic number of H and $\omega(H)$ is the supremum of the sizes of cliques in H. (A **clique** in this paper means a set of vertices, not necessarily maximal, which spans a complete subgraph.) For infinite graphs it is possible to define a stronger notion, in the spirit of [1, 5]:

Definition 1.1: A graph G is **strongly perfect** if every induced subgraph H of G has a coloring (C_i: $i \in I$) (meaning that the "colors" C_i are independent sets forming a partition of $V(H)$) and a clique K such that $K \cap C_i \neq \emptyset$ for all $i \in I$.

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In the finite case perfectness clearly implies strong perfectness, but this is false, in general, for infinite graphs. To see this, take G to be the disjoint union of cliques K_n of size n ($n < \omega$). But the result of Oellrich and Steffens [5] prompted the first author and V. Korman to make the following conjecture:

CONJECTURE 1.2: *If G is perfect and contains no infinite independent set, then G is strongly perfect.*

Henceforth we shall call a graph with no infinite independent set **narrow**. The Oellrich–Steffens result can be formulated as follows:

THEOREM 1.3: *A narrow incomparability graph is strongly perfect.*

(G is an **incomparability graph** if there exists a poset P on $V(G)$, such that $(x, y) \in E(G)$ if and only if x and y are incomparable in P. Theorem 1.3 is a "strong version" of Dilworth's theorem. It is proved in [5] for the countable case, but the same proof, using the theorem of [1] instead of its countable version, proves the full result.)

The outstanding special case of Conjecture 1.2 is that of comparability graphs. (A **comparability graph**, excuse the circuitous definition, is the complement of an incomparability graph....)

In [3] a stronger conjecture than Conjecture 1.2 was suggested, and we would like to re-state it here. First, we introduce the following terminology:

A **hypergraph** H is a pair $(V = V(H),\ E = E(H))$, where E is a set of finite subsets of V, such that $\bigcup E = V$. The elements of E are called "edges". A matching (i.e., a set of disjoint edges) in a hypergraph H is called **strongly maximal** if one cannot add to it k edges of H (k being a finite number) and delete fewer than k edges, and remain with a matching.

An **edge cover** of H is a set of edges whose union is $V(H)$. It is called **strongly minimal** if one cannot delete k edges from it and add fewer than k and remain with an edge cover.

CONJECTURE 1.4:

(a) *Every hypergraph contains a strongly maximal matching*

and

(b) *Every hypergraph contains a strongly minimal cover.*

Conjecture 1.2 follows from Conjecture 1.4(b) by a compactness argument used after applying Conjecture 1.4(b) to the hypergraph of independent sets of the graph in Conjecture 1.2.

There is no implication known between the two parts of the conjecture. Both parts are true for graphs [2], and thus Conjecture 1.2 is known for graphs G whose largest independent set is of size 2.

If true, Conjecture 1.4 will probably be difficult to prove—even the proof for graphs uses heavy machinery. A more modest aim is to prove Conjecture 1.2 for special classes of perfect graphs. The best known such classes, after those of comparability and incomparability graphs, are those of chordal graphs and their complements. In this paper we prove the conjecture for these two classes. Let us first recall the definition:

Definition: A graph G is called **chordal** if each cycle of length 4 or more in G contains a chord.

The two theorems which will be proved in this paper are:

THEOREM 1.5: *A narrow chordal graph is strongly perfect*

and:

THEOREM 1.6: *A narrow complement of a chordal graph is strongly perfect.*

Remark: The disjoint union of complete graphs of size n ($n < \omega$) shows that the narrowness condition is indeed necessary in Theorem 1.5, and the transitive closure of the infinite binary tree shows that it is necessary in Theorem 1.6.

Notation: Given a graph G and a subset A of $V(G)$, we write $G[A]$ for the subgraph of G induced by A, and $G - A$ for $G[V(G) \setminus A]$.

2. Narrow complements of chordal graphs are strongly perfect

In this section we prove Theorem 1.6. Since an induced subgraph of a chordal graph is chordal, it suffices to prove the following:

THEOREM 2.1: *Let G be a chordal graph with no infinite clique. Then there exists a partition of $V(G)$ into complete subgraphs (K_i: $i \in I$) and an independent set C meeting all K_i.*

The proof will use the notion of **tree representations**. A pair $(T, (T_v\colon v \in V))$, where T is a tree and $(T_v\colon v \in V)$ a collection of subtrees of T, is called a **tree representation** of a graph $G = (V, E)$ if $E =$

$\{(u,v): V(T_u) \cap V(T_v) \neq \emptyset\}$ (i.e. two vertices are connected in G if and only if the subtrees corresponding to them intersect). Our main tool is:

THEOREM 2.2 [4]: *A chordal graph with no infinite clique has a tree representation.*

Let $\mathcal{T} = (T, (T_v: v \in V))$ be a tree representation of a graph. Given a vertex x of T we write $\mathcal{T}_x = \{T_v: x \in V(T_v)\}$.

Whenever a tree representation $\mathcal{T} = (T, (T_v: v \in V))$ is considered, we shall associate with it a vertex $r = r(\mathcal{T})$ of T, which will play the role of the "root" of T. The choice of r is almost arbitrary: the only requirement we make is that

$$r \in V(T_v) \quad \text{for some minimal subtree } T_v, \quad \text{if such exists.} \tag{1.1}$$

("Minimality" here is with respect to containment.) For any vertex x of T different from r we write $p(x)$ for the vertex immediately preceding x in T, i.e. $p(x)$ is adjacent to x and nearer to r than x.

A choice of a root r for T induces a choice of a root $r(S)$ for every subtree S of T: (S) is the vertex of S nearest to r (possibly $r(S) = r$).

Another definition needed for the proof is:

Definition 2.3: Let T be a rooted tree with root r and let $A \subseteq V(T) \setminus \{r\}$. We write $T - A$ for the tree with root r whose vertex set is $V(T) \setminus A$ and whose edge set is defined by: $(x,y) \in E(T-A)$ if x precedes y in T (i.e. the path from r to y contains x) or vice versa, and all vertices on the path between x and y (in T) belong to A. If $\mathcal{T} = (T, (T_v: v \in V))$ is a tree representation, we write $\mathcal{T} - A$ for the system $(T - A, (T_v - (A \cap V(T_v)): v \in V))$.

Proof of Theorem 2.1: Let $\mathcal{T} = (T, (T_v: v \in V))$ be a tree representation of G. Since G has no infinite clique $\mathcal{T}$ does not contain an infinite descending sequence of subtrees, and hence we have:

ASSERTION 2.4: *Every T_v contains a minimal (with respect to containment) subtree T_u.*

In terms of $\mathcal{T}$, what Theorem 2.1 says is that there exists a subset Y of V and a choice of a vertex $v_y \in V(T_y)$ for each $y \in Y$ such that the trees T_y, $y \in Y$ are vertex disjoint, and the set $\{v_y: y \in Y\}$ meets all trees T_v. In fact, the task is simplified by the fact that a choice can be made in which $v_y = r(T_y)$ for each $y \in Y$.

The first step is to get rid of "redundancies" in $\mathcal{T}$, which is done as follows. We define inductively subsets U_α of V and subsets A_α of $V(T)$. Let $A_0 = U_0 = \emptyset$. Assume that A_β, U_β have been defined for $\beta < \alpha$ (where $\alpha \geq 1$). Let $\hat{A}_\alpha = \bigcup\{A_\beta: \beta < \alpha\}$, $\hat{U}_\alpha = \bigcup\{U_\beta: \beta < \alpha\}$. Let $\mathcal{T}_\alpha = (T, (T_v: v \in V \setminus \hat{U}_\alpha)) - \hat{A}_\alpha$. Define U_α to be the set of vertices $u \in V \setminus \hat{U}_\alpha$ such that $V(T_v) \setminus \hat{A}_\alpha \subseteq V(T_u) \setminus \hat{A}_\alpha$ for some $v \neq u$, $v \in V \setminus \hat{U}_\alpha$. (That is, U_α is the set of indices of non-minimal trees in the system $\mathcal{T}_\alpha$.) Let A_α be the set of vertices $x \in V(T) \setminus \hat{A}_\alpha$ such that $x \neq r(T_v)$ for all $v \in V \setminus \hat{U}_\alpha \setminus U_\alpha$. (Note that given a tree representation $(R, (R_z: z \in Z))$ with root r, the vertices $x \neq r$ which are not roots of any tree R_z are precisely those vertices x for which $\mathcal{R}_x \subseteq \mathcal{R}_{p(x)}$.) Note also that by (1.1) and Assertion 2.4, $r \notin A_\alpha$.

The definition terminates when $U_\theta = \emptyset$ for some $\theta > 1$ (this must clearly happen for some $\theta < |V|^+$). Let then $\mathcal{T}_\theta$ be defined as above and write: $\mathcal{T}_\theta = \mathcal{R} = (R, (R_z: z \in Z))$. Let H be the graph represented by $\mathcal{T}_\theta$.

Assertion 2.5: (a) *Every tree R_z is minimal with respect to containment, and* (b) *Every vertex in R is a root of some R_z.*

Proof: Part (a) is clear, since $U_\theta = \emptyset$. For the proof of (b) assume first that $\theta = \rho + 1$. Then $\mathcal{T}_\theta$ is obtained by deleting all non-roots (vertices in A_ρ) from $T - \hat{A}_\rho$ in the tree representation $(T, (T_v: v \in V \setminus \hat{U}_\theta)) - \hat{A}_\rho$, and hence each vertex in $T - A_\theta$ is a root in $\mathcal{T}_\theta$. For θ limit, since a vertex can be a root of only finitely many trees, each non-root x in $\mathcal{T}_\theta$ must have been a non-root in $(T, (T_v: v \in V \setminus \hat{U}_{\alpha+1})) - \hat{A}_\alpha$ for some $\alpha < \theta$, and then $x \in A_\alpha$, i.e. $x \notin V(R)$.

We first prove the desired result for $\mathcal{R}$:

Assertion 2.6: *There exists a subset Y of Z such that the trees R_y, $y \in Y$ are vertex-disjoint, and the set $\{r(R_y): y \in Y\}$ meets all trees R_z.*

Proof: By (1.1) and Assertion 2.4, $r = r(R_{y_0})$ for some $y_0 \in Z$. We choose inductively vertices y_α as follows. Assume that $\alpha > 0$ and that $(y_\beta: \beta < \alpha)$ have been chosen. Let $W_\alpha = \bigcup\{V(R_{y_\rho}): \rho < \alpha\}$. If $W_\alpha = V(R)$ then stop the process of definition, otherwise choose any vertex $t \in V(R) \setminus W_\alpha$ such that $p(t) \in W_\alpha$. Let y_α be such that $r(V_y) = t$ (such y_α exists, by Assertion 2.5(b)).

The process must terminate at some ordinal $\zeta < |V|^+$. Let $Y = \{y_\alpha: \alpha < \zeta\}$. By the choice of the y_α, the trees T_y, $y \in Y$ are disjoint. The way Y is constructed implies that $W_\zeta = V(R)$. Let $z \in Z$. Then $r(R_z) \in V(R_y)$ for some $y \in Y$. If $z \neq y$ then, by Assertion 2.5(a), $V(R_z) \not\subset V(R_y)$. Hence, there exists a vertex $x \in V(R_z) \setminus V(R_y)$ such that $p(x) \in V(R_y)$. Then $x \in V(R_u)$ for some $u \in Y$, and since $V(R_u) \cap V(R_y) = \emptyset$, there holds $x = r(R_u)$. Thus, R_z meets $\{r(y): y \in Y\}$, as required.

Since $V(R_v) \subseteq V(R_u)$ for $u \neq v$, in particular $V(R_v) \neq V(R_u)$, and hence for each $y \in Y$ there exists a unique $v = v(y) \in V$ such that $R_y = T_v - \hat{A}_\theta$. Let $W = \{v(y): y \in Y\}$. The proof of Theorem 2.1 will be complete if we prove:

ASSERTION 2.7:

(a) *The trees T_w, $w \in W$, are disjoint.*

(b) *Each T_v, $v \in V$, contains $r(T_w)$ for some $w \in W$.*

Proof: (a) Let $u, v \in W$, $u \neq v$. If $V(T_u) \cap V(T_w) \neq \phi$, then either $r(T_u) \in V(T_v)$ or $r(T_v) \in V(T_w)$, say $r(T_u) \in V(T_v)$. Since $u \in W$, we have $r(T_u) \notin \hat{A}_\theta$. But then $r(T_u) \in V(T_u - \hat{A}_\theta) \cap V(T_v - \hat{A}_\theta)$, contradicting the disjointness of the trees R_y, $y \in Y$.

(b) Let $v \in V$. If $v \in W$ then we are done. Otherwise, there exists α_1 such that $v \in U_{\alpha_1}$. Then there exists $u_1 \in V \setminus \hat{U}_{\alpha_1+1}$ such that $V(T_{u_1}) \setminus \hat{A}_{\alpha_1} \subseteq V(T_v) \setminus \hat{A}_{\alpha_1}$. Since, by the definition of the sets A_α, $r(T_{u_1}) \notin \hat{A}_{\alpha_1}$, we have $r(T_{u_1}) \in V(T_v)$. If $u_1 \in W$ then, again, we are done. If not, then there exists $\alpha_2 > \alpha_1$ such that $u_1 \in U_{\alpha_2}$. This means that there exists $u_2 \in V \setminus \hat{U}_{\alpha_2+1}$ such that $V(T_{u_2}) \setminus \hat{A}_{\alpha_2} \subseteq V(T_{u_1}) \setminus \hat{A}_{\alpha_2}$. By the choice of the sets A_α, we have $r(T_{u_2}) \notin \hat{A}_{\alpha_2}$, and hence, $r(T_{u_2}) \in V(T_{u_1}) \setminus \hat{A}_{\alpha_1} \subseteq V(T_v) \setminus \hat{A}_{\alpha_1}$. Continuing in this way, we obtain a sequence of subtrees T_{u_i} and ordinals α_i (where $u_0 = v$ and $\alpha_0 = 0$) such that for all $i < j$ there holds $r(T_{u_i}) \in V(T_{u_j}) \setminus \hat{A}_{\alpha_j}$. This means that the vertices u_i form a clique in G, and hence, by the assumption on G, that the sequence u_i is finite. Let u_k be the last element in the sequence. Then $u_k \in W$ and $r(T_{u_k}) \in V(T_v)$, which proves (b).

3. Narrow chordal graphs are strongly perfect

The main tool in the proof of Theorem 1.5, as in the finite case, is the following lemma:

LEMMA 3.1: *A minimal cut in a chordal graph is a clique.*

(A **cut** is a set of vertices whose deletion makes the graph disconnected.)

The proof of the lemma is the same as that of its finite version (see e.g. in Berge's book *Graphs and Hypergraphs*, Chapter 16. The lemma is originally due to Hajnal and Suranyi).

Another result which we shall need is:

THEOREM 3.2: *The complement of a bipartite graph is strongly perfect.*

This follows from Theorem 1.3, since a bipartite graph with sides A, B is a comparability graph, upon defining $x > y$ if $x \in A$, $y \in B$ and $(x, y) \in E$.

Definition 3.3: A **tree-expansion** of a graph G is a tree T with a root r, together with a choice of a subset X_v of $V(G)$ for every vertex v of T and a clique C_v in $G[X_v]$ for every non-leaf vertex v of T, which satisfy the following conditions:

(1) $X_r = V(G)$;

(2) If $\{u_i: i \in I\}$ is the set of sons of v in T, then $X_{u_i} = C_v \cup D_i$, where D_i is the union of certain connected components of $G[X_v] - C_v$, $D_i \cap D_j = \emptyset$ for $i \neq j$ and $\bigcup\{D_i: i \in I\} = X_v \setminus C_v$.

(3) $G[X_v]$ is the complement of a bipartite graph (i.e., X_v is the union of two cliques) for every leaf v of T.

If G has a finite tree expansion then it is called **finitely tree expandible** (f.t.e.).

Definition 3.4: A cut C **separates** two cliques A and B if there exist two points, one in A and the other in B, which are separated by C.

Definition 3.5: Let A, C be cliques. We say that A is C-**sociable** if either
(a) C is a cut, and $G - C$ has more than two connected components, or
(b) the component of $G - C$ meeting A is not contained in A. (Note that in (b) we do not insist that C is a cut—the component in question may be the whole graph $G - C$.)

LEMMA 3.6: *Let G be a non-f.t.e. chordal graph and let A,B be cliques in G which are not cuts. Then there exists a clique C such that either*

(1) *C separates A from B and both A and B are C-sociable*

or

(2) *$G - C$ has a component which does not meet $A \cup B$.*

Proof: If $A \cup B$ is a clique then take $C = A \cup B$. If not, then choose $x \in A$ and $y \in B$ which are not connected and let D be a minimal cut separating x from y. If (1) fails for D then, say, A is not D-sociable. Let $\mathcal{D}$ be the set of clique-cuts D separating A from B and for which A is not D-sociable.

ASSERTION 3.6A: *If no clique C satisfies (2), then there exists $D \in \mathcal{D}$ for which $A \cap D$ is minimal.*

Proof of the assertion: By Zorn's Lemma, it suffices to show that if $(D_\alpha\colon \alpha < \theta)$ is a sequence of cliques in $\mathcal{D}$ such that $(D_\alpha \cap A\colon \alpha < \theta)$ is strictly descending (i.e., $D_\alpha \cap A \subsetneq D_\beta \cap A$ whenever $\alpha > \beta$), then there exists $F \in \mathcal{D}$ with $F \cap A = \bigcap_{\alpha<\theta} D_\alpha \cap A$.

Let $\alpha > \beta$. Choose $v \in (D_\beta \setminus D_\alpha) \cap A$. Every vertex $u \in (D_\beta \setminus D_\alpha) \setminus A$ is connected to v, and since D_α separates A from B, it follows that $D_\alpha \setminus A$ separates u from B. Thus, $D_\alpha \setminus A$ separates $(D_\beta \setminus D_\alpha) \setminus A$ from B.

Let $F_1 = \bigcap_{\alpha<\theta} D_\alpha \cap A$ and let F_2 be the set of vertices $x \notin A$ which belong eventually to all cliques D_α (i.e., there exists $\beta < \theta$ such that $x \in D_\alpha$ for all $\alpha \geq \beta$). Finally, let $F = F_1 \cup F_2$. (If θ is a successor ordinal $\beta + 1$ then, clearly, $F = D_\beta$.) Clearly, F is a clique. We shall show that $F \in \mathcal{D}$.

SUB-ASSERTION: *Let $y \in V \setminus (A \cup F)$. If D_α separates A from y for all α (where possibly y belongs to D) then so does F.*

Proof of the sub-assertion: Let $x \in A \setminus F$. Let P be a path from x to y, and let z be the last vertex on P (starting from x) which belongs to some D_γ. Let $\delta > \gamma$. Take $v \in (D_\gamma \setminus D_\delta) \cap A$. Since D_δ separates v from y, it must contain some vertex on P between z and y (including endpoints). But by the choice of z this means that $z \in D_\delta$. Thus, $z \in D_\delta$ for all $\delta \geq \gamma$, meaning that $z \in F$.

Two corollaries follow from the sub-assertion:

(i) The component of $G - F$ meeting A is contained in A;

(ii) If $F \not\supseteq B$ then F separates A from B.

If $F \supseteq B$ then, by (i) and the fact that G is not f.t.e., $G - F$ has a component which misses A (and, of course, also B). But this means that F satisfies (2), a contradiction.

Thus $F \not\supseteq B$. By (i) and (ii), F belongs then to $\mathcal{D}$. This proves the assertion.

Let us return now to the proof of the lemma. Let D be as in the assertion. If $D \cup B$ is a clique (in particular, if $D \supseteq B$), then, since A is not D-sociable, and G is not f.t.e., $G - (D \cup B)$ must have a component missing A, and then (2) is satisfied. We may therefore assume that $D \cup B$ is not a clique.

Let H be the component of $G - D$ which meets B, and let $\tilde{G} = G[V(H) \cup D]$. Let I be a clique separating D from B in $\tilde{G}$. Suppose that A is not I-sociable. Then $I \in \mathcal{D}$, and by the minimality property of D this implies that $I \cap A = D \cap A$.

But I separates some vertex $x \in D \setminus I$ from B, and by the above $x \notin A$. Thus x either belongs to a component of $G - I$ not meeting $A \cup B$, or it belongs to the component meeting A. In either case A is I-sociable. By the negation assumption on (1) B is not I-sociable.

If T is a clique in $\tilde{G}$ such that $\tilde{G} - T$ has a component not meeting $B \cup D$, then $G - T$ has a component not meeting $A \cup B$. Hence, replacing G by $\tilde{G}$, A by B and B by D, we may apply Assertion 3.6a.

Let $\mathcal{K}$ be the set of clique-cuts K in $\tilde{G}$ separating B from D, and for which B is not K-sociable. Note that $I \in \mathcal{K}$, and thus $\mathcal{K} \neq \emptyset$. By Assertion 3.6a there exists $K \in \mathcal{K}$ for which $K \cap B$ is minimal.

Let L be the component of $\tilde{G} - K$ meeting D, and let $G^* = G[V(L) \cup K]$. If $K \cup D$ is a clique, then $G - (K \cup D)$ has a component not meeting $A \cup B$, or else (by the non-sociability of A and B with respect to D and K, respectively) G would be f.t.e. But this means that (2) holds, and thus we may assume that $K \cup D$ is not a clique. Let M be a clique in G^* which separates K from D. Then M separates A from B in G. By the minimality properties of K and D it follows (as before) that A and B are both M-sociable, and thus M satisfies (1).

THEOREM 3.7: *A narrow chordal graph is finitely tree expandible.*

Proof: Suppose that G is not f.t.e. We shall construct a sequence G_i $(i < \omega)$ of non-f.t.e. graphs, G_{i+1} being an induced subgraph of G_i . The graphs G_i $(i \geq 1)$ will be divided into two types, "free" and "besieged". In a free G_i there will be chosen a clique A_i, and in a besieged G_i two cliques will be chosen, A_i and B_i. These will be chosen inductively so that

(a) If G_i is free then A_i separates $V(G_i) \setminus A_i$ from $V(G) \setminus V(G_i)$.
(b) If G_i is besieged then $A_i \cup B_i$ separates $V(G_i) \setminus (A_i \cup B_i)$ from $V(G) \setminus V(G_i)$, and
(c) No A_i or B_i is a cut in G_i.

For each i we shall also define a set Z_i of vertices.

Let $G_0 = G$. Since G is not f.t.e. there exists in G a clique-cut C_0. Since G is narrow, $G - C_0$ has only finitely many connected components, and hence, for at least one of them, call it D, the graph $G_1 = G[C_0 \cup V(D)]$ is not f.t.e. We define G_1 as free, and let $A_1 = C_0$, $Z_1 = V(G) \setminus V(G_1)$.

Assume now that G_i, as well as A_i (and B_i, if G_i is besieged) are defined. If G_i is free then, since it is not f.t.e., $V(G_i) \setminus A_i$ is not a clique. Choose

$x, y \in V(G_i) \setminus A_i$ which are not connected, and a minimal cut C separating them. Choose a component D of $G_i - C$ such that $G_i[V(D) \cup C]$ is not f.t.e., and let $G_{i+1} = G_i[V(D) \cup C]$. If D meets A_i then define G_{i+1} as besieged and let $A_{i+1} = A_i$ and $B_{i+1} = C$. If $V(D) \cap A_i = \emptyset$ define G_{i+1} as free and let $A_{i+1} = C$. By the choice of x, y and D there exists a connected component F of $G_i - C$ different from D which contains a point outside $V(A_i) \cup V(C)$. Define $Z_{i+1} = V(F) \setminus (V(A_i) \cup V(C))$.

Assume that G_i is besieged. Apply Lemma 3.6 to G_i, A_i and B_i, and let C be a clique as in the lemma. If case (1) occurs, choose a component D of $G_i - C$ for which $G[V(D) \cup C]$ is not f.t.e., let $G_{i+1} = G[V(D) \cup C]$ and $A_{i+1} = C$. Let $B_{i+1} = A_i$ if D meets A_i, $B_{i+1} = B_i$ if D meets B_i (and let G_{i+1} be besieged in either case), and define G_{i+1} as free if $D \cap (A_i \cup B_i) = \emptyset$. In all these cases let $Z_{i+1} = V(F) \setminus (V(D) \cup A_i \cup B_i)$ for some component F of $G_i - C$ which is different from D and for which this set is non-empty (the existence of such F follows from the C-sociability of A_i and B_i).

Suppose next that case (2) occurs. Let F be a component of $G - C$ not meeting $A \cup B$. Let $G_{i+1} = G_1[V(F) \cup C]$, define G_{i+1} as free, and let $A_{i+1} = C$, $Z_{i+1} = \emptyset$.

Properties (a), (b) and (c) (mentioned at the beginning of this proof) are easily shown inductively. By (a), (b) and the choice of the sets Z_i, points from different Z_i's are not connected. Note that $Z_i = \emptyset$ only when G_{i+1} is free, in which case $Z_{i+1} \neq \emptyset$. Hence $Z_i \neq \emptyset$ infinitely often, implying the existence of an infinite independent set.

Proof of Theorem 1.6: Let G be a narrow chordal graph. By Theorem 3.7 it has a finite tree expansion, for which we shall use the notation of Definition 3.3 (i.e., a tree T, sets X_v and cliques C_v). The proof will proceed by induction on $|V(T)|$. If T consists of a single vertex, then G is complete and the theorem is obvious. So, assume that $|V(T)| > 1$, and let v be a leaf of T. Let $X = X_v$ and $Y = X_v \setminus C_u$, where u is the father of v. The graph $G' = G - Y$ has a tree expansion whose tree is $T - \{v\}$. By the induction hypothesis there exists a decomposition of G' into independent sets I_j ($j \in J$) and a clique K in G' which meets all I_j.

Let $J_1 = \{j \in J\colon I_j \cap X \setminus Y \neq \emptyset\}$ and $J_2 = J \setminus J_1$. Consider the following cases:

Case I. $|Y| \leq |J_2|$. Let $f\colon Y \to J_2$ be an injection. For each $j \in f[Y]$ let

$I'_j = I_j \cup \{f^{-1}(j)\}$, and for $j \in J \setminus f[Y]$ let $I'_j = I_j$. Since all neighbors of vertices from Y belong to X, the sets I'_j are independent, their union is V and, of course, they all meet K.

Case II. $|Y| > |J_2|$. Let F be a subset of Y of cardinality J_2, and let $g\colon F \to J_2$ be a bijection. Let $I'_j = I_j \cup \{g^{-1}(j)\}$ for each $j \in J_2$ and $I'_j = I_j$ for $j \in J_1$. To the system I'_j, $j \in J$, add all singletons $\{y\}$, $y \in Y \setminus F$. The resulting system of independent sets covers V, and it has a clique which meets all its members, namely X.

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